

SKEW CELL MODULES FOR DIAGRAM ALGEBRAS

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ABSTRACT. We provide an explicit construction of skew cell modules for diagram algebras.

INTRODUCTION

The representation theory of the symmetric group, \mathfrak{S}_r , is governed by its associated branching graph $\widehat{\mathfrak{S}}$ (also known as Young's lattice). The set of vertices on the r th level this graph, $\widehat{\mathfrak{S}}_r$, is given by the set of partitions of r ; the edges connect partitions which differ by a single node. The first few levels of this graph are depicted in Figure 1. Given λ a vertex on the r th level of this graph, there is an associated $\mathbb{Q}\mathfrak{S}_r$ -module, $S_r^{\mathbb{Q}}(\lambda)$, called a **Specht module**. This module has an integral basis indexed by the set of paths in the branching graph which begin at \emptyset and terminate at λ ; we refer to such paths as **standard tableaux** of shape λ . The Specht modules $\{S_r^{\mathbb{Q}}(\lambda) \mid \lambda \in \widehat{\mathfrak{S}}_r\}$ provide a complete set of simple $\mathbb{Q}\mathfrak{S}_r$ -modules.

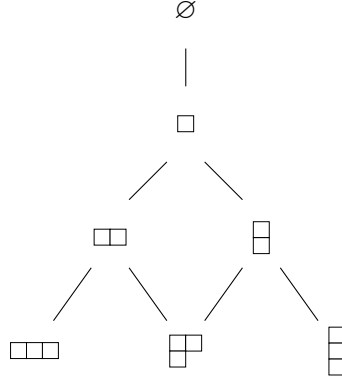


FIGURE 1. The first few levels of $\widehat{\mathfrak{S}}$.

For $1 \leq s \leq r$, we consider \mathfrak{S}_s as a subgroup of \mathfrak{S}_r under the embedding $\mathfrak{S}_s \subseteq \mathfrak{S}_{r-s} \times \mathfrak{S}_s \subseteq \mathfrak{S}_r$. In other words, \mathfrak{S}_s is the subgroup of permutations of the set $\{r-s+1, \dots, r-1, r\} \subseteq \{1, \dots, r\}$. Generalising the above, for any two vertices $\lambda \in \widehat{\mathfrak{S}}_{r-s}$ and $\nu \in \widehat{\mathfrak{S}}_r$, we may define the skew Specht module

$$S_s^{\mathbb{Q}}(\nu \setminus \lambda) := \text{Hom}_{\mathfrak{S}_{r-s}}(S_{r-s}^{\mathbb{Q}}(\lambda), \text{Res}_{\mathfrak{S}_{r-s}}^{\mathfrak{S}_r} S_r^{\mathbb{Q}}(\nu)).$$

The skew Specht module $S_s^{\mathbb{Q}}(\nu \setminus \lambda)$ has an integral basis indexed by the set of paths in the branching graph which begin at the vertex λ and terminate at the vertex ν ; we refer to such paths as **skew tableaux** of shape $\nu \setminus \lambda$.

Skew Specht modules carry the structure of simple modules for affine Hecke algebras [19] and play an important role in the graded representation theory of the symmetric groups and their affine analogues, see [17]. Most importantly for us, one obtains a particularly natural interpretation of the skew-Kostka and Littlewood–Richardson coefficients as the multiplicities

$$K_{\lambda, \mu}^{\nu} = \dim_{\mathbb{Q}} \text{Hom}_{\mathfrak{S}_s}(\text{ind}_{\mathfrak{S}_{\mu}}^{\mathfrak{S}_{\lambda}}(\mathbb{Q}), S_s^{\mathbb{Q}}(\nu \setminus \lambda)) \quad c_{\lambda, \mu}^{\nu} = \dim_{\mathbb{Q}} \text{Hom}_{\mathfrak{S}_s}(S_s^{\mathbb{Q}}(\mu), S_s^{\mathbb{Q}}(\nu \setminus \lambda))$$

respectively (here $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \dots$). We have a natural map $\varphi : \text{ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_s}(\mathbb{Q}) \rightarrow S_s^{\mathbb{Q}}(\mu)$ and so we can immediately deduce that $c_{\lambda,\mu}^\nu \leq K_{\lambda,\mu}^\nu$. In [12], James shows that $\text{Hom}_{\mathfrak{S}_s}(\text{ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_s}(\mathbb{Q}), S_s^{\mathbb{Q}}(\nu \setminus \lambda))$ has a basis indexed by the set of **semistandard** skew tableaux of shape $\nu \setminus \lambda$ and weight μ . James then shows that the homomorphisms which factor through the map φ are indexed by the subset of these skew tableaux which satisfy the **lattice permutation** property. This provides a concrete combinatorial interpretation of the coefficients $K_{\lambda,\mu}^\nu$ and $c_{\lambda,\mu}^\nu$.

The purpose of this article is to generalise the classical construction of skew Specht modules for the symmetric group to the setting of diagram algebras (such as the Brauer, walled Brauer, Temperley–Lieb, Birman–Murakami–Wenzl, and partition algebras) over a field \mathbb{F} . The representation theory of these diagram algebras is again controlled by their associated branching graphs. Each vertex λ of the branching graph labels a cell module $\Delta^{\mathbb{F}}(\lambda)$ and the paths in the branching graph index integral bases of these cell modules. Given two fixed points λ and ν in the branching graph, we provide an explicit construction of an associated **skew cell module** $\Delta^{\mathbb{F}}(\nu \setminus \lambda)$. We show that these skew cell modules possess integral bases indexed by skew tableaux (paths between the two fixed vertices in the graph) exactly as in the classical case. We expect the skew cell modules to play an important role in the graded representation theory of these algebras (which has only recently begun to be explored, see [13]) and provide a link between finite dimensional diagrammatic algebras and their affine analogues.

The partition algebra over the rational numbers, $P_s^{\mathbb{Q}}(n)$, will provide the motivating example of a diagram algebra throughout this paper. Precisely as in the classical case, one can define the multiplicities

$$\Gamma_{\lambda,\mu}^\nu = \dim_{\mathbb{Q}} \text{Hom}_{P_s^{\mathbb{Q}}(n)}(\text{ind}_{P_\mu^{\mathbb{Q}}(n)}^{P_s^{\mathbb{Q}}(n)}(\mathbb{Q}), \Delta_s^{\mathbb{Q}}(\nu \setminus \lambda)) \quad \bar{g}_{\lambda,\mu}^\nu = \dim_{\mathbb{Q}} \text{Hom}_{P_s^{\mathbb{Q}}(n)}(\Delta_s^{\mathbb{Q}}(\mu), \Delta_s^{\mathbb{Q}}(\nu \setminus \lambda))$$

(here $P_\mu^{\mathbb{Q}}(n) = P_{\mu_1}^{\mathbb{Q}}(n) \times P_{\mu_2}^{\mathbb{Q}}(n) \dots$) and $\bar{g}_{\lambda,\mu}^\nu \leq \Gamma_{\lambda,\mu}^\nu$ as before. In this paper we prove that the coefficients $\bar{g}_{\lambda,\mu}^\nu$ are equal to the **stable Kronecker coefficients**. These coefficients have been described as ‘perhaps the most challenging, deep and mysterious objects in algebraic combinatorics’ [18]. On the other hand, the coefficients $\Gamma_{\lambda,\mu}^\nu$ do not seem to have been studied anywhere in the literature. Motivated by the classical case, we ask the following questions:

- can one interpret the coefficients $\Gamma_{\lambda,\mu}^\nu$ and $\bar{g}_{\lambda,\mu}^\nu$ in terms of the combinatorics of skew-tableaux for the partition algebra?
- do there exist natural generalisations of the **semistandard** and **lattice permutation** conditions in this setting?
- do the coefficients $\Gamma_{\lambda,\mu}^\nu$ provide a first step towards understanding the stable Kronecker coefficients $\bar{g}_{\lambda,\mu}^\nu$?

We shall address these questions in an upcoming series of papers.

The paper is structured as follows. In Section 1 we recall the axiomatic framework for diagram algebras due to Enyang, Goodman, and Graber. We focus on the construction of Murphy bases of these diagram algebras which are encoded via the branching graph. In Section 2 we consider the restriction of a cell module down the tower of algebras; we construct filtrations of these restricted modules which will be essential in Sections 3 and 4. In Section 3 we recall the definitions of Jucys–Murphy elements and seminormal forms of diagram algebras, following Goodman–Graber and Mathas. We use the results of Section 2 in order to improve upon these results (these stronger statements will then be used in Section 4). Section 4 contains the main results of the paper. Given a cell module for a diagram algebra, we examine the action of certain Young subalgebras on the Murphy basis. We hence construct integral bases for the skew modules which are indexed by skew tableaux.

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1. DIAGRAM ALGEBRAS

For the remainder of the paper, we shall let R be an integral domain with field of fractions \mathbb{F} . In this section, we shall define diagram algebras and recall the construction of their Murphy bases, following [6]. We first recall the definition of a cellular algebra, as in [10].

Definition 1.1. Let R be an integral domain. A **cellular algebra** is a tuple $(A, *, \hat{A}, \triangleright, \mathscr{A})$ where

- (1) A is a unital R -algebra and $*$: $A \rightarrow A$ is an algebra involution, that is, an R -linear anti-automorphism of A such that $(x^*)^* = x$ for $x \in A$;
- (2) $(\hat{A}, \triangleright)$ is a finite partially ordered set, and $\text{Std}(\lambda)$, for $\lambda \in \hat{A}$, is a finite indexing set;
- (3) The set

$$\mathscr{A} = \{c_{\mathbf{st}}^\lambda \mid \lambda \in \hat{A} \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\},$$

is an R -basis for A , for which the following conditions hold:

- (a) Given $\lambda \in \hat{A}$, $\mathbf{t} \in \text{Std}(\lambda)$, and $a \in A$, there exist coefficients $r(a, \mathbf{t}, \mathbf{v}) \in R$, for $\mathbf{v} \in \text{Std}(\lambda)$, such that, for all $\mathbf{s} \in \text{Std}(\lambda)$,

$$c_{\mathbf{st}}^\lambda a \equiv \sum_{\mathbf{v} \in \text{Std}(\lambda)} r(a, \mathbf{t}, \mathbf{v}) c_{\mathbf{sv}}^\lambda \pmod{A^{\triangleright \lambda}}, \quad (1.1)$$

where $A^{\triangleright \lambda}$ is the R -module with basis

$$\{c_{\mathbf{st}}^\mu \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\mu) \text{ and } \mu \triangleright \lambda\}.$$

- (b) If $\lambda \in \hat{A}$ and $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$, then $(c_{\mathbf{st}}^\lambda)^* = (c_{\mathbf{ts}}^\lambda)$.

The tuple $(A, *, \hat{A}, \triangleright, \mathscr{A})$ is a **cell datum** for A . The basis \mathscr{A} is called a **cellular basis** of A .

We let $A^\mathbb{F}$ denote the algebra $A \otimes_R \mathbb{F}$. We say that $A^\mathbb{F}$ is a cellular algebra, with cell-datum $(A^\mathbb{F}, *, \hat{A}, \triangleright, \mathscr{A}^\mathbb{F})$, where the basis $\mathscr{A}^\mathbb{F}$ is given by $\{c_{\mathbf{st}}^\lambda \otimes 1_\mathbb{F} \mid \lambda \in \hat{A} \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\}$. When no confusion is possible, we denote an element $a \otimes 1_\mathbb{F} \in A^\mathbb{F}$ simply by $a \in A^\mathbb{F}$.

Definition 1.2. Let A be a cellular algebra over the integral domain R . Given $\lambda \in \hat{A}$, we define the **cell module** $\Delta^R(\lambda)$ to be the right A -module with R -basis

$$\{c_{\mathbf{st}}^\lambda + A^{\triangleright \lambda} \mid \mathbf{t} \in \text{Std}(\lambda)\},$$

for any fixed $\mathbf{s} \in \text{Std}(\lambda)$. We let $\Delta^\mathbb{F}(\lambda)$ denote the module $\Delta^R(\lambda) \otimes_R \mathbb{F}$.

Definition 1.3 ([7]). A cellular algebra, A , is said to be **cyclic cellular** if every cell module is cyclic as an A -module.

Definition 1.4. A cellular algebra A over the integral domain R is said to be **generically semisimple** if $A^\mathbb{F} = A \otimes_R \mathbb{F}$ is semisimple.

Remark 1.5. Throughout this paper we shall assume that all our algebras are cyclic cellular and generically semisimple.

1.1. Strongly coherent towers of cyclic cellular algebras. In this section, we recall the definitions and Murphy bases of strongly coherent towers of cyclic cellular algebras.

Definition 1.6. Let A be a cyclic cellular algebra over R . If M is a right A -module, a **cell-filtration** of M is a filtration by right A -modules

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M,$$

such that $M_i/M_{i-1} \cong \Delta^R(\lambda^{(i)})$ for some $\lambda^{(i)} \in \hat{A}$. We say that the filtration is **order preserving** if $\lambda^{(i)} \triangleright \lambda^{(i+1)}$ in \hat{A} for all $i \geq 1$.

Here and in the remainder of the paper, we will let $(A_k)_{k \geq 0}$ denote an increasing sequences

$$R = A_0 \subseteq A_1 \subseteq A_2 \cdots$$

of cyclic cellular algebras over an integral domain R . We assume that all the inclusions are unital and that the involutions are consistent; that is the involution on A_{k+1} , restricted to A_k , agrees with the involution on A_k .

Definition 1.7 ([8, 9]). The tower of cyclic cellular algebras $(A_k)_{k \geq 0}$ is **coherent** if the following conditions are satisfied:

- (1) For each $k \geq 0$ and each cell module $\Delta_k^R(\lambda)$ for $\lambda \in \widehat{A}_k$, the induced module $\text{Ind}_{A_k}^{A_{k+1}}(\Delta_k^R(\lambda))$ has a cell-filtration.
- (2) For each $k \geq 0$ and each cell module $\Delta_{k+1}^R(\mu)$ for $\mu \in \widehat{A}_{k+1}$, the restricted module $\text{Res}_{A_k}^{A_{k+1}}(\Delta_{k+1}^R(\mu))$ has a cell-filtration.

The tower is called **strongly coherent** if the cell-filtrations can be chosen to be order preserving.

Definition 1.8. Let $(A_k)_{k \geq 0}$ denote a strongly coherent tower of cyclic cellular algebras. We define the branching graph, \widehat{A} , as follows. The set of vertices on the k th level of \widehat{A} is given by the set \widehat{A}_k . Given $\lambda \in \widehat{A}_k$ and $\mu \in \widehat{A}_{k+1}$, there is an edge $\lambda \rightarrow \mu$ if and only if the module $\Delta_k^R(\lambda)$ appears in the cell-filtration of $\text{Res}_{A_k}^{A_{k+1}}(\Delta_{k+1}^R(\mu))$.

Definition 1.9. Given $\lambda \in \widehat{A}_{r-s}$, $\nu \in \widehat{A}_r$, we define a skew standard tableau of shape $\nu \setminus \lambda$ and degree s to be a path \mathbf{t} of the form

$$\lambda = \mathbf{t}(0) \rightarrow \mathbf{t}(1) \rightarrow \mathbf{t}(2) \rightarrow \cdots \rightarrow \mathbf{t}(s-1) \rightarrow \mathbf{t}(s) = \nu,$$

in other words \mathbf{t} is a path in the branching graph which begins at λ and terminates at ν . We let $\text{Std}_s(\nu \setminus \lambda)$ denote the set of all such paths; if $\lambda \in \widehat{A}_0$ then we let $\text{Std}_r(\nu) := \text{Std}_r(\nu \setminus \lambda)$.

Theorem 1.10 ([6]). Let $(A_k)_{k \geq 0}$ denote a strongly coherent tower of generically semisimple cyclic cellular algebras over R . Given $r \in \mathbb{Z}_{\geq 0}$, the algebra A_r has a cellular basis (referred to as the **Murphy basis**) indexed by paths in the branching graph as follows,

$$\{d_{\mathbf{s}}u_{\mathbf{t}} \mid \mathbf{s}, \mathbf{t} \in \text{Std}_r(\nu), \nu \in \widehat{A}_r\}. \quad (1.2)$$

Moreover, each basis element can be factorised in the following fashion,

$$d_{\mathbf{s}} = d_{\mathbf{s}(0) \rightarrow \mathbf{s}(1)} d_{\mathbf{s}(1) \rightarrow \mathbf{s}(2)} \cdots d_{\mathbf{s}(r-1) \rightarrow \mathbf{s}(r)} \quad u_{\mathbf{t}} = u_{\mathbf{t}(r-1) \rightarrow \mathbf{t}(r)} u_{\mathbf{t}(r-2) \rightarrow \mathbf{t}(r-1)} \cdots u_{\mathbf{t}(0) \rightarrow \mathbf{t}(1)}$$

where the branching coefficients, $d_{\mathbf{s}(k) \rightarrow \mathbf{s}(k+1)}$ (respectively $u_{\mathbf{t}(k) \rightarrow \mathbf{t}(k+1)}$) depend only on the vertices $\mathbf{s}(k) \in \widehat{A}_k$, $\mathbf{s}(k+1) \in \widehat{A}_{k+1}$ (respectively $\mathbf{t}(k) \in \widehat{A}_k$, $\mathbf{t}(k+1) \in \widehat{A}_{k+1}$).

Remark 1.11. By [6, Section 3.4], we have that $u_{\mathbf{t}} + A_r^{\triangleright \nu}$ is a non-zero element of $A_r^{\triangleright \nu} / A_r^{\triangleright \nu}$ for any $\mathbf{t} \in \text{Std}_r(\nu)$. Therefore by condition (3a) of Definition 1.1, we have that

$$\Delta_r^R(\nu) \cong \{d_{\mathbf{s}}u_{\mathbf{t}} + A_r^{\triangleright \nu} \mid \mathbf{t} \in \text{Std}_r(\nu)\} \cong \{u_{\mathbf{t}} + A_r^{\triangleright \nu} \mid \mathbf{t} \in \text{Std}_r(\nu)\}$$

as right A_r -modules, for any fixed $\mathbf{s} \in \text{Std}_r(\nu)$. We shall therefore use the simplified notation on the right-hand side of the above equation for the remainder of the paper.

Example 1.12 ([6]). Consider the chain of symmetric groups $(\mathfrak{S}_k)_{k \geq 0}$ under the obvious embedding. The branching graph $\widehat{\mathfrak{S}}$ is the usual Young graph, with $\widehat{\mathfrak{S}}_k$ equal to the set of partitions of k . There is an edge $\lambda \rightarrow \mu$ if μ is obtained from λ by adding a single node. The first few levels of this graph are depicted in Figure 1. The resulting cellular basis (as in Theorem 1.10) is the Murphy basis from [16].

Definition 1.13 (see [9]). We have two orderings on $\text{Std}_s(\nu \setminus \lambda)$, as follows. Let $\lambda \in \widehat{A}_{r-s}$, $\nu \in \widehat{A}_r$ and $\mathbf{s}, \mathbf{t} \in \text{Std}_s(\nu \setminus \lambda)$.

- We say that s precedes t in the **reverse lexicographic order** (denoted $s \preceq t$) if $s = t$, or if for the last index k such that $s(k) \neq t(k)$, we have $s(k) < t(k)$ in \widehat{A}_k .
- We say that s precedes t in the **dominance order** (denoted $s \leq t$) if $s(k) \leq t(k)$ in \widehat{A}_k for all $r - s \leq k \leq r$.

Clearly the latter ordering is a coarsening of the former.

Remark 1.14. In previous works on diagram algebras, [5, 6, 8, 9], the reverse lexicographic ordering has been used as the standard ordering on tableaux. In Section 4, we shall show that the reverse lexicographic ordering can be replaced with the (coarser) dominance ordering.

1.2. The Jones basic construction. First recall that an **essential idempotent** in an algebra A over a ring R is an element e such that $e^2 = \delta e$ for some non-zero $\delta \in R$.

Definition 1.15 ([6]). Let R be an integral domain with field of fractions \mathbb{F} and consider two towers of R -algebras with common multiplicative identity,

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \quad \text{and} \quad H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots. \quad (1.3)$$

Suppose that the two towers satisfy the following list of axioms:

- (1) There is an algebra involution $*$ on $\cup_k A_k$ such that $(A_k)^* = A_k$, and likewise, there is an algebra involution $*$ on $\cup_k H_k$ such that $(H_k)^* = H_k$.
- (2) $A_0 = H_0 = R$ and $A_1 = H_1$ (as algebras with involution).
- (3) For $k \geq 2$, A_k contains an essential idempotent e_{k-1} such that $e_{k-1}^* = e_{k-1}$ and $A_k / (A_k e_{k-1} A_k) \cong H_k$ as algebras with involution.
- (4) For $k \geq 1$, e_k commutes with A_{k-1} and $e_k A_k e_k \subseteq A_{k-1} e_k$.
- (5) For $k \geq 1$, $A_{k+1} e_k = A_k e_k$, and the map $x \mapsto x e_k$ is injective from A_k to $A_k e_k$.
- (6) For $k \geq 2$, $e_{k-1} \in A_{k+1} e_k A_{k+1}$.
- (7) $(H_k)_{k \geq 0}$ is a strongly coherent tower of cellular algebras.
- (8) For $k \geq 2$, $e_{k-1} A_k e_{k-1} A_k = e_{k-1} A_k$.
- (9) Each H_k is a cyclic cellular algebra.
- (10) For all $k \geq 0$, the algebra $A_k^{\mathbb{F}} := A_k \otimes_R \mathbb{F}$ is split semisimple.

In this case we say that the tower $(A_k)_{k \geq 0}$ is a tower of **diagram algebras** obtained by reflections from the tower of algebras $(H_k)_{k \geq 0}$.

The main theorem from [6] in which we are interested is the following.

Theorem 1.16 ([6]). Suppose that $(A_k)_{k \geq 0}$ is a tower of diagram algebras obtained by reflections from the tower $(H_k)_{k \geq 0}$. Let \widehat{H} denote the branching graph for $(H_k)_{k \geq 0}$ and let

$$\bar{d}_{\lambda \rightarrow \mu}^{(k)} \quad \bar{u}_{\lambda \rightarrow \mu}^{(k)}$$

denote the branching coefficients for $\lambda \in \widehat{H}_k$, $\mu \in \widehat{H}_{k+1}$ and $k \in \mathbb{Z}_{\geq 0}$.

In which case, the tower of algebras $(A_k)_{k \geq 0}$ is a strongly coherent tower of generically semisimple cyclic cellular algebras. Moreover, the branching graph \widehat{A} is obtained by reflections from \widehat{H} in the following fashion. The set, \widehat{A}_k , of vertices on level k is given by

$$\widehat{A}_k = \{(\lambda, l) \mid 0 \leq l \leq \lfloor k/2 \rfloor \text{ and } \lambda \in \widehat{H}_{k-2l}\}.$$

Given $(\lambda, l) \in \widehat{A}_k$ and $(\mu, m) \in \widehat{A}_{k+1}$, there is an edge $(\lambda, l) \rightarrow (\mu, m)$ only if $m \in \{l, l+1\}$; moreover

- $(\lambda, l) \rightarrow (\mu, l)$ in \widehat{A} if and only if $\lambda \rightarrow \mu$ in \widehat{H} ;
- $(\lambda, l) \rightarrow (\mu, l+1)$ in \widehat{A} if and only if $\mu \rightarrow \lambda$ in \widehat{H} .

The branching factors for the tower $(A_k)_{k \geq 0}$ can be given in terms of the branching factors of $(H_k)_{k \geq 0}$ in the following fashion:

- $d_{(\lambda, l) \rightarrow (\mu, l)}^{(k+1)} = \bar{d}_{\lambda \rightarrow \mu}^{(k+1-2l)} e_{k-1}^{(l)}$;

- $u_{(\lambda,l) \rightarrow (\mu,l)}^{(k+1)} = \bar{u}_{\lambda \rightarrow \mu}^{(k+1-2l)} e_k^{(l)};$
- $d_{(\lambda,l) \rightarrow (\mu,l+1)}^{(k+1)} = \bar{u}_{\mu \rightarrow \lambda}^{(k-2l)} e_{k-1}^{(l)};$
- $u_{(\lambda,l) \rightarrow (\mu,l+1)}^{(k+1)} = \bar{d}_{\mu \rightarrow \lambda}^{(k-2l)} e_k^{(l+1)};$

for $(\lambda, l) \in \hat{A}_k$ and $(\mu, m) \in \hat{A}_{k+1}$ and $k \in \mathbb{Z}_{\geq 0}$.

Remark 1.17. We note that the map $\hat{A}_k \rightarrow \cup_{0 \leq l \leq \lfloor k/2 \rfloor} \hat{H}_{k-2l}$ given by $(\lambda, l) \rightarrow \lambda$ is a bijection. Therefore, when no confusion is possible, we identify the indexing labels under this bijection.

Example 1.18. Examples of algebras which fit into the above framework include (cyclotomic) Brauer, walled Brauer, partition, Temperley–Lieb, and Birman–Murakami–Wenzl algebras.

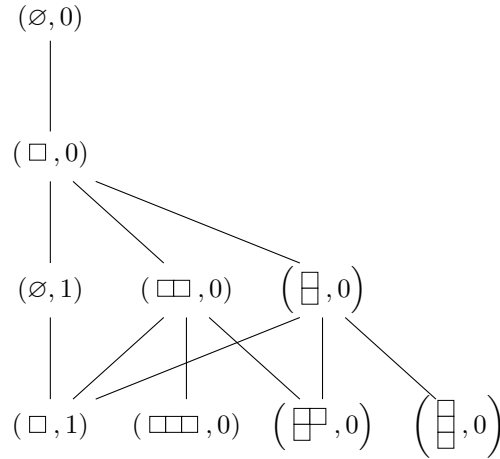


FIGURE 2. The first few levels of the branching graph, \hat{B} , of the Brauer algebras.

Example 1.19 ([6]). Let B_k denote the Brauer algebra on k strands defined in [3]. Wenzl [22] showed that the Brauer algebras form a tower of diagram algebras, $(B_k)_{k \geq 0}$, obtained ‘by reflections’ from the (tower of) symmetric groups (see Example 1.12). Therefore, the branching graph of the Brauer algebras $(B)_{k \geq 0}$ is that obtained from $\hat{\mathfrak{S}}$ by reflections. This graph has vertices on level k given by the set

$$\{(\lambda, l) \mid 0 \leq l \leq \lfloor k/2 \rfloor \text{ and } \lambda \in \hat{\mathfrak{S}}_{k-2l}\}.$$

There is an edge $(\lambda, l) \rightarrow (\mu, l)$ if the partition μ is obtained from the partition λ by adding a single node. There is an edge $(\lambda, l) \rightarrow (\mu, l+1)$ if the partition μ is obtained from the partition λ by removing a single node. The first few levels of graph are depicted in Figure 2.

2. FILTRATIONS OF RESTRICTED MODULES

For the remainder of this paper, we shall assume that $(A_k)_{k \geq 0}$ is a strongly coherent tower of generically semisimple cyclic cellular algebras. The following small technical lemma allows us to reduce the problem of constructing submodules (in particular, skew modules) to the semisimple case.

Lemma 2.1. Let B^R (respectively A^R) denote an R -algebra and let $B^{\mathbb{F}}$ (respectively $A^{\mathbb{F}}$) denote the algebra $B^R \otimes_R \mathbb{F}$ (respectively $A^R \otimes_R \mathbb{F}$). Suppose that $\iota_1 : A^R \rightarrow B^R$ is an embedding of R -algebras. Let M^R denote a right B^R -module with R -basis $\{m_1, \dots, m_k\}$ and let N^R denote an R -submodule of M^R with R -basis $\{m_1, \dots, m_j\} \subseteq \{m_1, \dots, m_k\}$. We let ι_2 denote the obvious linear inclusion of R -modules $\iota_2 : N^R \rightarrow M^R$ and let $M^{\mathbb{F}} = M^R \otimes_R \mathbb{F}$, $N^{\mathbb{F}} = N^R \otimes_R \mathbb{F}$.

Assume that the subspace $N^{\mathbb{F}} \subseteq M^{\mathbb{F}}$ is an $A^{\mathbb{F}}$ -submodule. Then N^R carries the structure of an A^R -module, with action given by

$$N^R \times A^R \rightarrow N^R; \quad (n, a) \mapsto \iota_2(n)\iota_1(a),$$

for $n \in N^R, a \in A^R$.

Proof. Let $a \in A$ and $m_i \in N^R \subseteq M^R$, where $1 \leq i \leq j$. Since $\iota_1(a) \in B^R$ and $\iota_2(m_i) \in N^R$, we have

$$\iota_2(m_i)\iota_1(a) = \sum_{l=1}^k r_l m_l,$$

where $r_l \in R$ for $1 \leq l \leq k$. Hence

$$(\iota_2(m_i)\iota_1(a)) \otimes 1_{\mathbb{F}} = \sum_{l=1}^k (r_l \otimes 1_{\mathbb{F}})(m_l \otimes 1_{\mathbb{F}}) \in M^{\mathbb{F}}.$$

Therefore $r_l = 0$ whenever $j < l \leq k$. Hence $\iota_2(m_i)\iota_1(a) \in N^R$ for all $1 \leq i \leq j$, as required. \square

Definition 2.2. Let $\nu \in \widehat{A}_r$ and $\lambda \in \widehat{A}_{r-s}$ be such that $\text{Std}_s(\nu \setminus \lambda) \neq \emptyset$. We denote

$$M_{s,r}^R(\lambda, \nu) = \text{Span}_R \{u_{\mathbf{t}} + A_r^{\triangleright \nu} \mid \mathbf{t} \in \text{Std}_r(\nu) \text{ and } \mathbf{t}(r-s) \triangleright \lambda\}$$

and

$$U_{s,r}^R(\lambda, \nu) = \text{Span}_R \{u_{\mathbf{t}} + A_r^{\triangleright \nu} \mid \mathbf{t} \in \text{Std}_r(\nu) \text{ and } \mathbf{t}(r-s) \triangleright \lambda\}.$$

We set $M_{s,r}^{\mathbb{F}}(\lambda, \nu) = M_{s,r}^R(\lambda, \nu) \otimes_R \mathbb{F}$ and $U_{s,r}^{\mathbb{F}}(\lambda, \nu) = U_{s,r}^R(\lambda, \nu) \otimes_R \mathbb{F}$.

The following lemma states that for a subalgebra $A_{r-s}^{\mathbb{F}} \subseteq A_r^{\mathbb{F}}$ under the tower embedding, restriction of cell modules from $A_r^{\mathbb{F}}$ to $A_{r-s}^{\mathbb{F}}$ is compatible with the cellular basis. Recall the conventions from [Remark 1.17](#).

Lemma 2.3. Let $\nu \in \widehat{A}_r$ and $\lambda \in \widehat{A}_{r-s}$, where $\text{Std}_s(\nu \setminus \lambda) \neq \emptyset$. Then $U_{s,r}^{\mathbb{F}}(\lambda, \nu) \subseteq M_{s,r}^{\mathbb{F}}(\lambda, \nu) \subseteq \Delta_r^{\mathbb{F}}(\nu)$ as $A_{r-s}^{\mathbb{F}}$ -modules. Moreover $M_{s,r}^{\mathbb{F}}(\lambda, \nu)/U_{s,r}^{\mathbb{F}}(\lambda, \nu)$ is isomorphic as an $A_{r-s}^{\mathbb{F}}$ -module to a direct sum of $\sharp \text{Std}_s(\nu \setminus \lambda)$ copies of $\Delta_{r-s}^{\mathbb{F}}(\lambda)$.

Proof. We proceed by induction on the dominance order. Assume that λ is maximal in \widehat{A}_{r-s} subject to the condition that $\text{Std}_s(\nu \setminus \lambda) \neq \emptyset$. For $\mu \in \widehat{A}_{r-s}$, we let $E_{r-s}^{\mu} \in A_{r-s}^{\mathbb{F}} \subseteq A_r^{\mathbb{F}}$ denote the central idempotent which projects onto the simple $A_{r-s}^{\mathbb{F}}$ -module $\Delta_{r-s}^{\mathbb{F}}(\mu)$. Let $\mathbf{t} \in \text{Std}_r(\nu)$, where $\mathbf{t}(r-s) = \lambda$ and suppose that $\mu \in \widehat{A}_{r-s}$, where $\mu \triangleright \lambda$. By the cellularity of A_{r-s} (and our assumption that $\mu \triangleright \lambda$) we have

$$u_{\mathbf{t}_{[0,r-s]}} E_{r-s}^{\mu} \in A_{r-s}^{\triangleright \lambda} \otimes_R \mathbb{F}.$$

Thus the maximality of λ gives

$$u_{\mathbf{t}} \sum_{\mu \triangleright \lambda} E_{r-s}^{\mu} = \sum_{\mu \triangleright \lambda} u_{\mathbf{t}_{[r-s,r]}} u_{\mathbf{t}_{[0,r-s]}} E_{r-s}^{\mu} \in A_r^{\triangleright \nu} \otimes_R \mathbb{F} \quad (2.1)$$

Again by the cellularity of A_{r-s} , we have that

$$u_{\mathbf{t}} E_{r-s}^{\lambda} = u_{\mathbf{t}_{[r-s,r]}} u_{\mathbf{t}_{[0,r-s]}} E_{r-s}^{\lambda} = u_{\mathbf{t}} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}_{r-s}(\sigma)}} r_{\mathbf{u}, \mathbf{v}} u_{\mathbf{t}_{[r-s,r]}} (d_{\mathbf{u}}^* u_{\mathbf{v}}).$$

Since $\sum_{\mu \triangleright \lambda} E_{r-s}^{\mu}$ acts as the identity on the ideal $A_{r-s}^{\triangleright \lambda} \otimes_R \mathbb{F}$, to which the terms of the form $d_{\mathbf{u}}^* u_{\mathbf{v}}$ in the above sum belong, the relation $\sum_{\mu \triangleright \lambda} E_{r-s}^{\mu} = 0$ gives

$$0 = \sum_{\mu \triangleright \lambda} u_{\mathbf{t}} E_{r-s}^{\mu} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}_{r-s}(\sigma)}} r_{\mathbf{u}, \mathbf{v}} u_{\mathbf{t}_{[r-s,r]}} (d_{\mathbf{u}}^* u_{\mathbf{v}}).$$

However, by equation (2.1), we have

$$\sum_{\mu \triangleright \lambda} u_{\mathbf{t}} E_{r-s}^{\mu} \equiv 0 \pmod{A_r^{\triangleright \nu} \otimes_R \mathbb{F}}.$$

Hence $u_{\mathbf{t}} E_{r-s}^{\lambda} = u_{\mathbf{t}} \pmod{A_r^{\triangleright \nu} \otimes_R \mathbb{F}}$. Therefore as an $A_{r-s}^{\mathbb{F}}$ -module,

$$\text{Span}_{\mathbb{F}} \{u_{\mathbf{t}} + A_r^{\triangleright \nu} \otimes_R \mathbb{F} \mid \mathbf{t} \in \text{Std}_r(\nu) \text{ and } \mathbf{t}(r-s) = \lambda\}$$

is isomorphic to a direct sum of $\sharp \text{Std}_{r-s}(\nu \setminus \lambda)$ copies of $\Delta_{r-s}^{\mathbb{F}}(\lambda)$, as required.

Next, we relax the assumption on the maximality of λ and let $\mathbf{t} \in \text{Std}_r(\nu)$, where $\mathbf{t}(r-s) = \lambda$. As before, we may write

$$u_{\mathbf{t}} E_{r-s}^{\lambda} = u_{\mathbf{t}} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}_{r-s}(\sigma)}} r_{\mathbf{u}, \mathbf{v}} u_{\mathbf{t}[r-s, r]} (d_{\mathbf{u}}^* u_{\mathbf{v}})$$

where

$$0 = \sum_{\mu \triangleright \lambda} u_{\mathbf{t}} E_{r-s}^{\mu} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}_{r-s}(\sigma)}} r_{\mathbf{u}, \mathbf{v}} u_{\mathbf{t}[r-s, r]} (d_{\mathbf{u}}^* u_{\mathbf{v}}). \quad (2.2)$$

However, by the inductive hypothesis,

$$\Delta_r^{\mathbb{F}}(\nu) \sum_{\mu \triangleright \lambda} E_{r-s}^{\mu} = U_{s,r}^{\mathbb{F}}(\lambda, \nu)$$

is a right $A_{r-s}^{\mathbb{F}}$ -submodule of $\Delta_r^{\mathbb{F}}(\nu)$. Hence the rightmost sum in equation (2.2) can be rewritten as

$$\sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}_s(\sigma)}} r_{\mathbf{u}, \mathbf{v}} u_{\mathbf{t}[r-s, r]} d_{\mathbf{u}}^* u_{\mathbf{v}} \equiv \sum_{\substack{\mathbf{s} \in \text{Std}_r(\nu) \\ \mathbf{s}(r-s) \triangleright \lambda}} r_{\mathbf{s}} u_{\mathbf{s}} \pmod{A_r^{\triangleright \nu} \otimes_R \mathbb{F}}.$$

Therefore,

$$u_{\mathbf{t}} E_{r-s}^{\lambda} = u_{\mathbf{t}} + \sum_{\{\mathbf{s} \mid \mathbf{s}(r-s) \triangleright \lambda\}} r_{\mathbf{s}} u_{\mathbf{s}} \pmod{A_r^{\triangleright \nu} \otimes_R \mathbb{F}}.$$

Therefore $M_{r,s}^{\mathbb{F}}(\lambda, \nu)/U_{r,s}^{\mathbb{F}}(\lambda, \nu)$ is isomorphic as an $A_{r-s}^{\mathbb{F}}$ -module to a direct sum of $\sharp \text{Std}_s(\nu \setminus \lambda)$ copies of $\Delta_{r-s}^{\mathbb{F}}(\lambda)$. The result follows by induction. \square

By Lemmas 2.1 and 2.3, we immediately deduce that there exists a A_{r-s} -module structure on $M_{s,r}^R(\lambda, \nu)/U_{s,r}^R(\lambda, \nu)$. This is the first step towards providing a cell-filtration of $M_{s,r}^R(\lambda, \nu)/U_{s,r}^R(\lambda, \nu)$ as an A_{r-s} -module.

Corollary 2.4. Let $\nu \in \widehat{A}_r$ and $\lambda \in \widehat{A}_{r-s}$ be such that $\text{Std}_s(\nu \setminus \lambda) \neq \emptyset$. Then $U_{s,r}^R(\lambda, \nu) \subseteq M_{s,r}^R(\lambda, \nu) \subseteq \Delta_r^R(\nu)$ as A_{r-s} -modules.

The following technical lemma will be useful in the proof of Proposition 2.6, below.

Lemma 2.5. Let $0 \leq s \leq r$, $\lambda \in \widehat{A}_{r-s}$ and $\nu \in \widehat{A}_r$. Let $\mathbf{t} \in \text{Std}_r(\nu)$, where $\mathbf{t}(r-s) = \lambda$. Let $a \in A_{r-s}$ and suppose that

$$u_{\mathbf{t}} a \equiv \sum_{\mathbf{s} \in \text{Std}_{r-s}(\lambda)} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}[r-s, r]} u_{\mathbf{s}} + \sum_{\substack{\mathbf{z} \in \text{Std}_r(\nu) \\ \mathbf{z}[r-s, r] \triangleright \mathbf{t}[r-s, r]}} r_{\mathbf{z}} u_{\mathbf{z}} \pmod{A_r^{\triangleright \nu}}, \quad (2.3)$$

where the scalars $r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) \in R$ are the structure constants for $\Delta_{r-s}^R(\lambda)$ determined by

$$u_{\mathbf{t}_{[0, r-s]}} a = \sum_{\mathbf{s} \in \text{Std}_{r-s}(\lambda)} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{s}} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{p}, \mathbf{q} \in \text{Std}_{r-s}(\sigma)}} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{p}, \mathbf{q}) d_{\mathbf{p}}^* u_{\mathbf{q}}. \quad (2.4)$$

Then the rightmost sum in equation (2.3) satisfies the relation

$$\sum_{\substack{\mathbf{z} \in \text{Std}_r(\nu) \\ \mathbf{z}_{[r-s, r]} \triangleright \mathbf{t}_{[r-s, r]}}} r_{\mathbf{z}} u_{\mathbf{z}} \equiv \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{p}, \mathbf{q} \in \text{Std}_{r-s}(\sigma)}} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{p}, \mathbf{q}) u_{\mathbf{t}_{[r-s, r]}} d_{\mathbf{p}}^* u_{\mathbf{q}} \pmod{A_r^{\triangleright \nu}}.$$

Proof. Multiplying both sides of equation (2.4) by $u_{\mathbf{t}_{[r-s, r]}}$ on the left gives:

$$u_{\mathbf{t}} a = u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{t}_{[0, r-s]}} a = \sum_{\mathbf{s} \in \text{Std}_{r-s}(\lambda)} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}} + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{p}, \mathbf{q} \in \text{Std}_{r-s}(\sigma)}} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{p}, \mathbf{q}) u_{\mathbf{t}_{[r-s, r]}} d_{\mathbf{p}}^* u_{\mathbf{q}}.$$

Since $\mathbf{t}(r-s) = \mathbf{s}(r-s)$ for $\mathbf{s} \in \text{Std}_{r-s}(\lambda)$, we have that $u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}}$ is actually the element of the cellular basis of A_r labelled by the concatenation of the tableaux $\mathbf{t}_{[r-s, r]}$ and \mathbf{s} . The result follows. \square

Finally, we conclude this section by demonstrating that the A_{r-s} -module $M_{s,r}^R(\lambda, \nu)/U_{s,r}^R(\lambda, \nu)$ has a filtration in which every factor is isomorphic to $\Delta_{r-s}^R(\lambda)$. This filtration is given by the dominance order on the skew tableaux of shape $\nu \setminus \lambda$.

Proposition 2.6. Let $\lambda \in \widehat{A}_{r-s}$ and $\nu \in \widehat{A}_r$. Given $\mathbf{t} \in \text{Std}_r(\nu)$ and $a \in A_{r-s}$, we have that

$$u_{\mathbf{t}} a \equiv \sum_{\mathbf{s} \in \text{Std}_{r-s}(\lambda)} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}} + \sum_{\substack{\mathbf{z} \in \text{Std}_r(\nu) \\ \mathbf{z}_{[r-s, r]} \triangleright \mathbf{t}_{[r-s, r]}}} r_{\mathbf{z}} u_{\mathbf{z}} \pmod{A_r^{\triangleright \nu}}.$$

where $r_{\mathbf{z}}, r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) \in R$. Moreover, the coefficients $r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s})$ are the structure constants for $\Delta_{r-s}^R(\lambda)$ determined by

$$u_{\mathbf{t}_{[0, r-s]}} a \equiv \sum_{\mathbf{s} \in \text{Std}_{r-s}(\lambda)} r(a; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{s}} \pmod{A_{r-s}^{\triangleright \lambda}}.$$

Proof. The statement holds for $s = 1$ and $r \in \mathbb{Z}_{\geq 0}$ by [9, Proposition 2.18]. Assume by induction that the statement holds for $1 \leq s \leq r$, we shall show that the result therefore holds for $s + 1$. Assume that $a' \in A_{r-s-1}$ and $\mathbf{t}(r-s) = \lambda$. By the $s = 1$ case we have that

$$\begin{aligned} u_{\mathbf{t}_{[0, r-s]}} a' &= \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) = \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{s}} + \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) \triangleright \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{s}} \\ &\quad + \sum_{\substack{\sigma \triangleright \lambda \\ \mathbf{p}, \mathbf{q} \in \text{Std}_{r-s}(\sigma)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{p}, \mathbf{q}) d_{\mathbf{p}}^* u_{\mathbf{q}}. \end{aligned} \tag{2.5}$$

where the first two (of the three) terms in equation (2.5) can be grouped together to be the first term in equation (2.4); the final term in equation (2.5) is equal to the final term in equation (2.4). As $a' \in A_{r-s-1} \subset A_{r-s}$, our inductive assumption implies that $u_{\mathbf{t}} a'$ is of the form required in equation (2.3). Therefore we can multiply both sides of equation (2.5) by $u_{\mathbf{t}_{[r-s, r]}}$ on the left and apply Lemma 2.5 in order to obtain

$$\begin{aligned} u_{\mathbf{t}} a' &\equiv \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) = \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}} + \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) \triangleright \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}} \\ &\quad + \sum_{\substack{\mathbf{z} \in \text{Std}_r(\nu) \\ \mathbf{z}_{[r-s, r]} \triangleright \mathbf{t}_{[r-s, r]}}} r_{\mathbf{z}} u_{\mathbf{z}} \pmod{A_r^{\triangleright \nu}}. \end{aligned} \tag{2.6}$$

By Corollary 2.4 and our assumption that $a \in A_{r-s-1}$, the final sum on the right hand side of equation (2.6) is over $\mathbf{z} \in \text{Std}_{r-s}(\nu)$ such that $\mathbf{z}(r-s-1) \triangleright \mathbf{t}(r-s-1)$ and can therefore be

expressed as:

$$\sum_{\substack{\mathbf{z} \in \text{Std}_{r-s}(\nu) \\ \mathbf{z}_{[r-s-1, r]} \triangleright \mathbf{t}_{[r-s-1, r]}}} r_{\mathbf{z}} u_{\mathbf{z}}$$

modulo $A_r^{\triangleright \nu}$. Rewritten as above, the final sum on the right hand side of equation (2.6) can be subsumed into the second sum on the right-hand side of equation (2.6) thereby giving

$$u_{\mathbf{t}} a' \equiv \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) = \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{t}_{[r-s, r]}} u_{\mathbf{s}} + \sum_{\substack{\mathbf{z} \in \text{Std}_{r-s}(\nu) \\ \mathbf{z}_{[r-s-1, r]} \triangleright \mathbf{t}_{[r-s-1, r]}}} r'_{\mathbf{z}} u_{\mathbf{z}} \pmod{A_r^{\triangleright \nu}}. \quad (2.7)$$

Now, by the inductive hypothesis the scalars $r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s})$, for $\mathbf{s} \in \text{Std}_{r-s}(\lambda)$ such that $\mathbf{s}(r-s-1) = \mathbf{t}(r-s-1)$, in equation (2.7) are the structure constants for the cell-modules of the algebra A_{r-s-1} ; that is the coefficients $r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s})$ are determined by the following equation

$$u_{\mathbf{t}_{[0, r-s-1]}} a' \equiv \sum_{\substack{\mathbf{s} \in \text{Std}_{r-s}(\lambda) \\ \mathbf{s}(r-s-1) = \mathbf{t}(r-s-1)}} r(a'; \mathbf{t}_{[0, r-s]}, \mathbf{s}) u_{\mathbf{s}_{[r-s-1, 0]}} \pmod{A_{r-s-1}^{\triangleright \mathbf{t}(r-s-1)}}.$$

This completes the proof of the proposition. \square

3. JUCYS–MURPHY ELEMENTS FOR DIAGRAM ALGEBRAS

We recall the definition and first properties of families of Jucys–Murphy elements for diagram algebras. The action of Jucys–Murphy elements on cell modules for diagram algebras was first considered systematically in Goodman–Grabner [8, 9]; motivated by work of Mathas [15]. In this section, we use Proposition 2.6 to strengthen the results of [8, 9] by replacing the reverse lexicographic order on skew tableaux with the dominance order on skew tableaux.

Definition 3.1. Let $(A_k)_{k \geq 0}$ be a strongly coherent tower of cellular algebras over an integral domain, R . We say that a family of elements $\{L_k : k \geq 1\}$, is an **additive family of Jucys–Murphy elements** if the following conditions hold.

- (1) For all $k \geq 1$, $L_k \in A_k$, L_k is invariant under the involution of A_k , and L_k commutes with A_{k-1} . In particular, $L_i L_j = L_j L_i$ for all $1 \leq i \leq j \leq k$.
- (2) For all $k \geq 1$ and $\lambda \in \widehat{A}_k$, there exists $d(\lambda) \in R$ such that $L_1 + \cdots + L_k$ acts as the scalar $d(\lambda)$ on the cell module $\Delta_k^R(\lambda)$. For $\lambda \in \widehat{A}_0$, we let $d(\lambda) = 0$.

Example 3.2. The group algebras of symmetric groups, Temperley–Lieb, Brauer, walled Brauer, and partition algebras all possess additive families of Jucys–Murphy elements.

Definition 3.3. Let $(A_k)_{k \geq 0}$ be a strongly coherent tower of cellular algebras over an integral domain, R . We say that a family of elements $\{L_k : k \geq 1\}$, is a **multiplicative family of Jucys–Murphy elements** if the following conditions hold.

- (1) For all $k \geq 1$, L_k is an invertible element of A_k , L_k is invariant under the involution $*$, and L_k commutes with A_{k-1} . In particular, $L_i L_j = L_j L_i$ for all $1 \leq i \leq j \leq k$.
- (2) For all $k \geq 1$ and $\lambda \in \widehat{A}_k$, there exists $d(\lambda) \in R$ such that $L_1 \cdots L_k$ acts as the scalar $d(\lambda)$ on the cell module $\Delta_k^R(\lambda)$. For $\lambda \in \widehat{A}_0$, we let $d(\lambda) = 1$.

Example 3.4. The Birman–Murakami–Wenzl algebra possess a multiplicative family of Jucys–Murphy elements.

In [9, Proposition 3.7] and [9, Proposition 3.6], Goodman and Grabner show that Jucys–Murphy elements act upper triangularly with respect to the reverse lexicographic order on skew tableaux. The main ingredient in their proof of is an analogue of Proposition 2.6, obtained by replacing the dominance order on skew tableaux with the weaker reverse lexicographic order on skew tableaux. Therefore Proposition 2.6 allows us to strengthen [9, Proposition 3.6] and [9,

Proposition 3.7] respectively by replacing the reverse lexicographic order on skew tableaux with the dominance order on skew tableaux (Definition 1.13). Therefore the subsequence applications of [9] (see for example, [5, 20, 21]) can be strengthened by replacing the reverse lexicographic order of [5] (and the miscellaneous orders used in [20, 21]) with the dominance order on skew tableaux.

Proposition 3.5. Suppose that $\{L_k : k \geq 1\}$ is an additive family of Jucys–Murphy elements for the tower $(A_k)_{k \geq 0}$.

- (1) For $k \geq 1$ and $\lambda \in \widehat{A}_k$, let $d(\lambda) \in R$ be such that $L_1 + \cdots + L_k$ acts by the scalar $d(\lambda)$ on the cell module $\Delta_k^R(\lambda)$. Then for all $k \geq 1$, $\lambda \in \widehat{A}_k$, $\mathbf{t} \in \text{Std}_k(\lambda)$, and $1 \leq j \leq k$, we have

$$u_{\mathbf{t}}^\lambda L_j = c_{\mathbf{t}}(j) u_{\mathbf{t}}^\lambda + \sum_{\mathbf{s}_{[j-1,k]} \triangleright \mathbf{t}_{[j-1,k]}} r_{\mathbf{s}} u_{\mathbf{s}}^\lambda \pmod{A_r^{\triangleright \lambda}}, \quad (3.1)$$

for some elements $r_{\mathbf{s}} \in R$ (depending on j and \mathbf{t}), with $c_{\mathbf{t}}(j) = d(\mathbf{t}(j)) - d(\mathbf{t}(j-1))$. Here, the order \triangleright is the dominance order on skew tableaux.

- (2) For each $k \geq 1$, $L_1 + \cdots + L_k$ is in the center of A_k .

Proof. One can replace all references to [9, Proposition 2.18] with references to Proposition 2.6 in the proof of [9, Proposition 3.7] \square

Proposition 3.6. Suppose that $\{L_k : k \geq 1\}$ is a multiplicative family of Jucys–Murphy elements for the tower $(A_k)_{k \geq 0}$.

- (1) For $k \geq 1$ and $\lambda \in \widehat{A}_k$, let $d(\lambda) \in R$ be such that $L_1 \cdots L_k$ acts by the scalar $d(\lambda)$ on the cell module $\Delta_k^R(\lambda)$. Then for all $k \geq 1$, $\lambda \in \widehat{A}_k$, $\mathbf{t} \in \text{Std}_k(\lambda)$, and $1 \leq j \leq k$, we have

$$u_{\mathbf{t}}^\lambda L_j = c_{\mathbf{t}}(j) u_{\mathbf{t}}^\lambda + \sum_{\mathbf{s}_{[j-1,k]} \triangleright \mathbf{t}_{[j-1,k]}} r_{\mathbf{s}} u_{\mathbf{s}}^\lambda \pmod{A_r^{\triangleright \lambda}}, \quad (3.2)$$

for some elements $r_{\mathbf{s}} \in R$ (depending on j and \mathbf{t}), with $c_{\mathbf{t}}(j) = \frac{d(\mathbf{t}(j))}{d(\mathbf{t}(j-1))}$. Here, the order \triangleright is the dominance order on skew tableaux.

- (2) For each $k \geq 1$, $L_1 \cdots L_k$ is in the center of A_k .

Proof. One can replace all references to [9, Proposition 2.18] with references to Proposition 2.6 in the proof of [9, Proposition 3.6]. \square

Definition 3.7 ([15]). Suppose that the map $\cup_{\lambda \in \widehat{A}_k} \text{Std}_k(\lambda) \rightarrow R^k$ given by $\mathbf{t} \mapsto (c_{\mathbf{t}}(j))_{1 \leq j \leq k}$ is injective for all $k \geq 1$. In this case, the Jucys–Murphy elements are said to satisfy the separation condition.

Definition 3.8. Let $(A_k)_{k \geq 0}$ be a strongly coherent tower of cellular algebras over R and let $(L_k)_{k \geq 1}$ be a set of additive or multiplicative Jucys–Murphy elements satisfying the separation condition. Given $\lambda \in \widehat{A}_k$ and $\mathbf{t} \in \text{Std}_k(\lambda)$, we define an element of the algebra $A_k^{\mathbb{F}}$ as follows

$$F_{\mathbf{t}} = \prod_{1 \leq i \leq k} \prod_{\substack{\mathbf{u} \in \text{Std}_k(\rho) \\ c_{\mathbf{u}}(i) \neq c_{\mathbf{t}}(i)}} \frac{L_i - c_{\mathbf{u}}(i)}{c_{\mathbf{t}}(i) - c_{\mathbf{u}}(i)},$$

where the product is taken over all $(\rho, k) \in \widehat{A}_k$. We let $f_{\mathbf{t}} = u_{\mathbf{t}} F_{\mathbf{t}}$ and $f_{\mathbf{st}} = F_{\mathbf{s}} d_{\mathbf{s}} u_{\mathbf{t}} F_{\mathbf{t}}$ for any pair $\mathbf{s}, \mathbf{t} \in \text{Std}_k(\lambda)$.

Proposition 3.9 (See [15, Section 3], [8, Section 3]). Let $(A_k)_{k \geq 0}$ be a strongly coherent tower of cyclic cellular algebras over R equipped with a set of additive or multiplicative Jucys–Murphy elements satisfying the separation condition.

(1) For $k \geq 1$, the set of paths from the zeroth to the k th level of the branching graph indexes a full set of pairwise orthogonal idempotents in $A_k^{\mathbb{F}}$. In other words

$$F_s F_t = \delta_{st} F_s \quad \sum_{\substack{s \in \text{Std}_k(\lambda) \\ \lambda \in \widehat{A}_k}} F_s = 1_{A_k^{\mathbb{F}}},$$

for $s \in \text{Std}_k(\lambda)$ and $t \in \text{Std}_k(\mu)$ and $\lambda, \mu \in \widehat{A}_k$. For $\lambda \in \widehat{A}_k$, we have idempotents,

$$1_k^{\triangleright \lambda} = \sum_{\substack{s \in \text{Std}_k(\mu) \\ \mu \triangleright \lambda}} F_s \quad 1_k^{\triangleright \lambda} = \sum_{\substack{s \in \text{Std}_k(\mu) \\ \mu \triangleright \lambda}} F_s$$

which act as the identity on the ideals $A_k^{\triangleright \lambda} \otimes_R \mathbb{F}$ and $A_k^{\triangleright \lambda} \otimes_R \mathbb{F}$ respectively. For $\lambda \in \widehat{A}_k$, the idempotent E_k^λ which acts by projection onto the simple $A_k^{\mathbb{F}}$ -module $\Delta_k^{\mathbb{F}}(\lambda)$ is equal to

$$E_k^\lambda = \sum_{s \in \text{Std}_k(\lambda)} F_s.$$

(2) If $t \in \text{Std}_k(\lambda)$ for $\lambda \in \widehat{A}$, then we have that

$$f_t = u_t + \sum_{\substack{u \in \text{Std}_k(\lambda) \\ u \triangleright t}} r_u u_t,$$

for scalars $r_u \in \mathbb{F}$.

- (3) $\{f_t \mid t \in \text{Std}_k(\lambda)\}$ is an \mathbb{F} -basis for $\Delta^{\mathbb{F}}(\lambda)$.
- (4) $f_t L_i = c_t(i) f_t$ for all $t \in \text{Std}_k(\lambda)$ and $i = 1, \dots, k$.
- (5) $f_s F_t = \delta_{st} f_s$ for all $s, t \in \text{Std}_k(\lambda)$.
- (6) $\langle f_s, f_t \rangle = \delta_{st} \langle f_s, f_s \rangle$ for all $s, t \in \text{Std}_k(\lambda)$.

Therefore the basis in (3) is a **seminormal basis** in the sense of [9] and is unique up to a choice of scaling factors in \mathbb{F} .

4. SKEW CELL MODULES FOR DIAGRAM ALGEBRAS

For the remainder of this paper, we shall assume that $(A_k)_{k \geq 0}$ is a tower of diagram algebras equipped with a family of Jucys–Murphy elements. In this section, we construct skew cell modules for diagram algebras and provide integral bases of these modules indexed by skew tableaux. Given $\lambda \in \widehat{A}_{r-s}$ and $\nu \in \widehat{A}_r$, we let $A_{s,r}^{\triangleright \nu \setminus \lambda}$ denote the R -subspace of A_r spanned by

$$\{d_s u_t \mid s, t \in \text{Std}_r(\mu), \mu \triangleright \nu\} \cup \{u_t \mid t \in \text{Std}_r(\nu), t(r-s) \triangleright \lambda\}, \quad (4.1)$$

and we let $A_{s,r} = A_{r-s} \times A_s \subseteq A_r$. We extend this notation to $A_{s,r}^{\mathbb{F}} = A_{s,r} \otimes_R \mathbb{F}$.

Lemma 4.1. If $\lambda \in \widehat{A}_{r-s}$ and $\nu \in \widehat{A}_r$, then:

- (1) $M_{s,r}^{\mathbb{F}}(\lambda, \nu) \subseteq \text{Res}_{A_{s,r}^{\mathbb{F}}}^{A_r^{\mathbb{F}}}(\Delta_r^{\mathbb{F}}(\nu))$ is an inclusion of $A_{s,r}^{\mathbb{F}}$ -modules.
- (2) $M_{s,r}^R(\lambda, \nu) \subseteq \text{Res}_{A_{s,r}^R}^{A_r^R}(\Delta_r^R(\nu))$ is an inclusion of $A_{s,r}^R$ -modules.

Proof. We first consider (1). We note that we have already proven the inclusion (1) on the level of $A_{r-s}^{\mathbb{F}}$ -modules; it remains to check the inclusion holds on the level of $A_{s,r}^{\mathbb{F}}$ -modules. We claim that

$$M_{s,r}^{\mathbb{F}}(\lambda, \nu) = \Delta_r^{\mathbb{F}}(\nu) 1_{r-s}^{\triangleright \lambda}. \quad (4.2)$$

Note that the idempotent $1_{r-s}^{\triangleright \lambda}$ is central in $A_{s,r}^{\mathbb{F}}$ and therefore the right-hand side of the equation carries the structure of an $A_{s,r}^{\mathbb{F}}$ -module. Hence to prove point (1), it is enough to show that the relation (4.2) holds. Given $\lambda \in \widehat{A}_{r-s}$ and $\nu \in \widehat{A}_r$, we define

$$\overline{M}_{s,r}^R(\lambda, \nu) = M_{s,r}^R(\lambda, \nu) / U_{s,r}^R(\lambda, \nu) \quad \overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) = \overline{M}_{s,r}^R(\lambda, \nu) \otimes_R \mathbb{F}.$$

By [Corollary 2.4](#), the space $\overline{M}_{s,r}^R(\lambda, \nu)$ carries the structure of an A_{r-s} -module. Moreover, by [Proposition 2.6](#), any total refinement of the dominance order on skew tableaux gives rise to a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = \overline{M}_{s,r}^R(\lambda, \nu) \quad (4.3)$$

of $\overline{M}_{s,r}^R(\lambda, \nu)$ by A_{r-s} -submodules, where

$$k = \sharp \text{Std}_s(\nu \setminus \lambda) \quad \text{and} \quad N_i/N_{i-1} \cong \Delta_{r-s}^R(\lambda) \quad \text{for } 1 \leq i \leq k. \quad (4.4)$$

By (4.3) and (4.4), we have that

$$\overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) E_{r-s}^\lambda = \overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) \quad (4.5)$$

as $A_{r-s}^{\mathbb{F}}$ -modules. In other words, for each $\mathbf{t} \in \text{Std}_r(\nu)$ such that $\mathbf{t}(r-s) = \lambda$, there exist $r_{\mathbf{st}} \in \mathbb{F}$ whereby

$$u_{\mathbf{t}} E_{r-s}^\lambda = \sum_{\substack{\mathbf{s} \in \text{Std}_r(\nu) \\ \mathbf{s}(r-s) = \lambda}} r_{\mathbf{st}} u_{\mathbf{s}} + \sum_{\substack{\mathbf{s} \in \text{Std}_r(\nu) \\ \mathbf{s}(r-s) \triangleright \lambda}} r_{\mathbf{st}} u_{\mathbf{s}} \quad \text{mod } A_r^{\triangleright \nu} \otimes_R \mathbb{F} \quad (4.6)$$

and

$$\overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) = \text{Span}_{\mathbb{F}} \{ u_{\mathbf{t}} E_{r-s}^\lambda + A_r^{\triangleright \nu \setminus \lambda} \otimes_R \mathbb{F} \mid \mathbf{t}(r-s) = \lambda \}. \quad (4.7)$$

We now consider the module $\Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda$. We note that we can identify

$$E_{r-s}^\lambda = \sum_{\mathbf{t} \in \text{Std}_{r-s}(\lambda)} F_{\mathbf{t}} = \sum_{\substack{\mu \in \widehat{A}_r \\ \mathbf{t} \in \text{Std}_r(\mu) \\ \mathbf{t}(r-s) = \lambda}} F_{\mathbf{t}} \in A_r^{\mathbb{F}}$$

by [Proposition 3.9](#). Therefore, in terms of the seminormal basis of the cell module,

$$\Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda = \text{Span}_{\mathbb{F}} \{ u_{\mathbf{s}} F_{\mathbf{s}} \mid \mathbf{s} \in \text{Std}_r(\nu) \} E_{r-s}^\lambda = \text{Span}_{\mathbb{F}} \{ f_{\mathbf{s}} \mid \mathbf{s} \in \text{Std}_r(\nu), \mathbf{s}(r-s) = \lambda \} \quad (4.8)$$

Hence, as a $A_{r-s}^{\mathbb{F}}$ -module, $\Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda$ decomposes as $\sharp \text{Std}_s(\nu \setminus \lambda)$ copies of $\Delta_{r-s}^{\mathbb{F}}(\lambda)$. We observe that equation (4.7) and equation (4.8) imply

$$\dim_{\mathbb{F}} (\Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda) = \dim_{\mathbb{F}} (\overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu)). \quad (4.9)$$

Having shown the equality of dimensions in equation (4.9), we may now prove the claim, equation (4.2), by induction on the dominance order. If λ is maximal we have that $U_{s,r}^{\mathbb{F}}(\lambda, \nu) = 0$ and so

$$\overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) E_{r-s}^\lambda = M_{s,r}^{\mathbb{F}}(\lambda, \nu) E_{r-s}^\lambda.$$

Therefore by equation (4.5), we have

$$M_{s,r}^{\mathbb{F}}(\lambda, \nu) E_{r-s}^\lambda = \text{Span}_{\mathbb{F}} \{ u_{\mathbf{t}} E_{r-s}^\lambda + A_r^{\triangleright \nu} \otimes_R \mathbb{F} \mid \mathbf{t}(r-s) = \lambda \}.$$

as an $A_{r-s}^{\mathbb{F}}$ -submodule of $\Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda$. Now, by equation (4.9), we have

$$\overline{M}_{s,r}^{\mathbb{F}}(\lambda, \nu) \cong M_{s,r}^{\mathbb{F}}(\lambda, \nu) = \Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\lambda,$$

as required. Now by our inductive assumption (on the dominance order on \widehat{A}_{r-s}), the space

$$U_{s,r}^{\mathbb{F}}(\lambda, \nu) = \sum_{\substack{\mu \in \widehat{A}_{r-s} \\ \mu \triangleright \lambda}} \Delta_r^{\mathbb{F}}(\nu) E_{r-s}^\mu = \Delta_r^{\mathbb{F}}(\nu) 1_{r-s}^{\triangleright \lambda}$$

is an $A_{s,r}^{\mathbb{F}}$ -module. We note that $E_{r-s}^\lambda = 1_{r-s}^{\triangleright \lambda} - 1_{r-s}^{\triangleright \lambda}$ and therefore by (4.6) and (4.7), we immediately obtain equation (4.2), by induction. This completes the proof of (1). Point (2) now follows immediately by [Lemma 2.1](#). \square

Theorem 4.2. Let $\lambda \in \widehat{A}_{r-s}$, $\nu \in \widehat{A}_r$, and let \mathbf{t}^λ be any maximal element in the dominance ordering on $\text{Std}_{r-s}(\lambda)$. The R -module

$$\Delta_s^R(\nu \setminus \lambda) = \text{Span}_R \{u_{\mathbf{u}}u_{\mathbf{t}^\lambda} + A_{s,r}^{\triangleright \nu \setminus \lambda} \mid \mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)\}$$

carries the structure of an A_s -module under the identification $1 \times A_s \subseteq A_{r-s} \times A_s \subseteq A_r$.

Proof. Let $\Delta_s^\mathbb{F}(\nu \setminus \lambda) = \Delta_s^R(\nu \setminus \lambda) \otimes_R \mathbb{F}$. It will suffice to show that

$$\Delta_s^\mathbb{F}(\nu \setminus \lambda) \subseteq \overline{M}_{s,r}^\mathbb{F}(\lambda, \nu) \quad (4.10)$$

as a module for $1 \times A_s^\mathbb{F} \subseteq A_{r-s}^\mathbb{F} \times A_s^\mathbb{F} \subseteq A_r^\mathbb{F}$. We know that $u_{\mathbf{t}^\lambda}F_{\mathbf{t}^\lambda} = u_{\mathbf{t}^\lambda}$ modulo $A_{r-s}^{\triangleright \lambda}$ and therefore, by Proposition 2.6, we conclude that for any $\mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)$ there exist scalars

$$\{r_{\mathbf{v}\mathbf{u}} \in \mathbb{F} \mid \mathbf{v} \in \text{Std}_s(\nu \setminus \lambda) \text{ and } \mathbf{v} \triangleright \mathbf{u}\},$$

which depend only on \mathbf{v} and \mathbf{u} , such that

$$u_{\mathbf{u}}u_{\mathbf{t}^\lambda}F_{\mathbf{t}^\lambda} \equiv u_{\mathbf{u}}u_{\mathbf{t}^\lambda} + \sum_{\substack{\mathbf{v} \in \text{Std}_s(\nu \setminus \lambda) \\ \mathbf{v} \triangleright \mathbf{u}}} r_{\mathbf{v}\mathbf{u}}u_{\mathbf{v}}u_{\mathbf{t}^\lambda} \pmod{A_{s,r}^{\triangleright \nu \setminus \lambda} \otimes_R \mathbb{F}}. \quad (4.11)$$

Hence the two sets

$$\{u_{\mathbf{u}}u_{\mathbf{t}^\lambda}F_{\mathbf{t}^\lambda} + A_{s,r}^{\triangleright \nu \setminus \lambda} \otimes_R \mathbb{F} \mid \mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)\} \quad \text{and} \quad \{u_{\mathbf{u}}u_{\mathbf{t}^\lambda} + A_{s,r}^{\triangleright \nu \setminus \lambda} \otimes_R \mathbb{F} \mid \mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)\} \quad (4.12)$$

span the same \mathbb{F} -module. Moreover, the change of basis matrix between the two bases in (4.12) is uni-triangular with respect to any total refinement of the dominance ordering on standard tableaux. By inverting this change of basis matrix we obtain scalars

$$\{r'_{\mathbf{u}\mathbf{v}} \in \mathbb{F} \mid \mathbf{v} \in \text{Std}_s(\nu \setminus \lambda) \text{ and } \mathbf{v} \triangleright \mathbf{u}\},$$

which depend only on \mathbf{v} and \mathbf{u} , such that

$$u_{\mathbf{u}}u_{\mathbf{t}^\lambda} \equiv u_{\mathbf{u}}u_{\mathbf{t}^\lambda}F_{\mathbf{t}^\lambda} + \sum_{\substack{\mathbf{v} \in \text{Std}_s(\nu \setminus \lambda) \\ \mathbf{v} \triangleright \mathbf{u}}} r'_{\mathbf{u}\mathbf{v}}u_{\mathbf{v}}u_{\mathbf{t}^\lambda}F_{\mathbf{t}^\lambda} \pmod{A_{s,r}^{\triangleright \nu \setminus \lambda} \otimes_R \mathbb{F}}. \quad (4.13)$$

Therefore, $\Delta_s^\mathbb{F}(\nu \setminus \lambda) = \overline{M}_{s,r}^\mathbb{F}(\lambda, \nu)F_{\mathbf{t}^\lambda}$ and hence equation (4.10) holds. The result now follows by Lemma 2.1. \square

For the purposes of book-keeping, we now collect together equation (4.1) and Theorem 4.2 in the following definition.

Definition 4.3. Given $\lambda \in \widehat{A}_{r-s}$, $\nu \in \widehat{A}_r$, we define the associated skew cell module, $\Delta_s^R(\nu \setminus \lambda)$, for A_s as follows:

$$\Delta_s^R(\nu \setminus \lambda) = \text{Span}_R \{u_{\mathbf{u}}u_{\mathbf{t}^\lambda} + A_{s,r}^{\triangleright \nu \setminus \lambda} \mid \mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)\}$$

where \mathbf{t}^λ is any maximal element in the dominance ordering on $\text{Std}_{r-s}(\lambda)$ and $A_{s,r}^{\triangleright \nu \setminus \lambda}$ denotes the R -subspace of A_r spanned by $\{d_s u_{\mathbf{t}} \mid \mathbf{s}, \mathbf{t} \in \text{Std}_r(\mu), \mu \triangleright \nu\} \cup \{u_{\mathbf{t}} \mid \mathbf{t} \in \text{Std}_r(\nu), \mathbf{t}(r-s) \triangleright \lambda\}$.

We now consider the decomposition of skew cell modules over the field of fractions, \mathbb{F} . We let $\Delta_s^\mathbb{F}(\nu \setminus \lambda) = \Delta_s^R(\nu \setminus \lambda) \otimes_R \mathbb{F}$.

Theorem 4.4. Given $\lambda \in \widehat{A}_{r-s}$, $\mu \in \widehat{A}_s$, $\nu \in \widehat{A}_r$, we have that

$$\text{Hom}_{A_{r-s}^\mathbb{F} \times A_s^\mathbb{F}}(\Delta_{r-s}^\mathbb{F}(\lambda) \boxtimes \Delta_s^\mathbb{F}(\mu), \text{Res}_{A_{r-s}^\mathbb{F} \times A_s^\mathbb{F}}^{A_r^\mathbb{F}}(\Delta_s^\mathbb{F}(\nu))) \cong \text{Hom}_{A_s^\mathbb{F}}(\Delta_s^\mathbb{F}(\mu), \Delta_s^\mathbb{F}(\nu \setminus \lambda)).$$

Proof. Any $\varphi \in \text{Hom}_{A_{r-s}^{\mathbb{F}} \times A_s^{\mathbb{F}}}(\Delta_{r-s}^{\mathbb{F}}(\lambda) \boxtimes \Delta_s^{\mathbb{F}}(\mu), \text{Res}_{A_{r-s}^{\mathbb{F}} \times A_s^{\mathbb{F}}}^{A_r^{\mathbb{F}}}(\Delta_s^{\mathbb{F}}(\nu)))$ is determined by

$$\varphi(f_{\mathbf{t}^\lambda} \times f_{\mathbf{t}^\mu}) = \sum_{\mathbf{s} \in \text{Std}_r(\nu)} a_{\mathbf{s}} u_{\mathbf{s}} \mod A_r^{\triangleright \nu}$$

for some $a_{\mathbf{s}} \in \mathbb{F}$. Moreover $F_{\mathbf{t}^\lambda} \cdot \varphi(f_{\mathbf{t}^\lambda} \times f_{\mathbf{t}^\mu}) = \varphi(f_{\mathbf{t}^\lambda} \times f_{\mathbf{t}^\mu})$ and therefore by equation (4.13) and the remarks immediately afterward, we conclude that φ is determined by

$$\varphi(f_{\mathbf{t}^\lambda} \times f_{\mathbf{t}^\mu}) = \sum_{\bar{\mathbf{s}} \in \text{Std}_s(\nu \setminus \lambda)} b_{\bar{\mathbf{s}}} u_{\bar{\mathbf{s}}} u_{\mathbf{t}^\lambda} \mod A_r^{\triangleright \nu \setminus \lambda} \quad (4.14)$$

for some $b_{\bar{\mathbf{s}}} \in \mathbb{F}$. By definition of the module $\Delta_s^{\mathbb{F}}(\nu \setminus \lambda)$, it is clear that any homomorphism $\psi \in \text{Hom}_{A_s^{\mathbb{F}}}(\Delta_s^{\mathbb{F}}(\mu), \Delta_s^{\mathbb{F}}(\nu \setminus \lambda))$ is determined by

$$\psi(f_{\mathbf{t}^\mu}) = \sum_{\mathbf{u} \in \text{Std}_s(\nu \setminus \lambda)} c_{\mathbf{u}} u_{\mathbf{u}} u_{\mathbf{t}^\lambda} \mod A_r^{\triangleright \nu \setminus \lambda} \quad (4.15)$$

for some $c_{\mathbf{u}} \in \mathbb{F}$. One can now apply the idempotent $F_{\mathbf{t}^\mu}$ to both sides of equation (4.14) and (4.15) and the result follows. \square

5. THE PARTITION ALGEBRA AND THE STABLE KRONECKER COEFFICIENTS

Recall that a **partition** λ is defined to be a weakly decreasing sequence of non-negative integers. We define the **degree** of the partition to be the sum $|\lambda| = \lambda_1 + \dots + \lambda_\ell$. Given λ a partition and $n \in \mathbb{N}$ sufficiently large we let

$$\lambda_{[n]} = (n - |\lambda|, \lambda_1, \lambda_2, \dots, \lambda_\ell).$$

Given λ a partition of degree $r - s$, μ a partition of degree s and ν a partition of degree less than or equal to r , we define the Kronecker coefficients to be the multiplicities

$$g_{\lambda_{[n]}, \mu_{[n]}}^{\nu_{[n]}} = \dim_{\mathbb{Q}} \text{Hom}_{\mathfrak{S}_n}(S^{\mathbb{Q}}(\lambda_{[n]}) \otimes S^{\mathbb{Q}}(\mu_{[n]}), S^{\mathbb{Q}}(\nu_{[n]})). \quad (5.1)$$

for n sufficiently large. As we let n increase, the sequence of coefficients obtained

- (1) is weakly increasing, in other words $g_{\lambda_{[n]}, \mu_{[n]}}^{\nu_{[n]}} \leq g_{\lambda_{[n+1]}, \mu_{[n+1]}}^{\nu_{[n+1]}}$ for all n ;
 - (2) and for some $N \in \mathbb{N}$ the sequence stabilises and we obtain $\bar{g}_{\lambda, \mu}^{\nu} = g_{\lambda_{[N+k]}, \mu_{[N+k]}}^{\nu_{[N+k]}}$ for all $k \geq 0$.
- The limiting values, $\bar{g}_{\lambda, \mu}^{\nu}$, are known as the **stable Kronecker coefficients**. This is illustrated in the following example.

Example 5.1. We have the following tensor products of Specht modules:

$$\begin{aligned} n = 2 & \quad \mathbf{S}(1^2) \otimes \mathbf{S}(1^2) = \mathbf{S}(2) \\ n = 3 & \quad \mathbf{S}(2, 1) \otimes \mathbf{S}(2, 1) = \mathbf{S}(3) \oplus \mathbf{S}(2, 1) \oplus \mathbf{S}(1^3) \\ n = 4 & \quad \mathbf{S}(3, 1) \otimes \mathbf{S}(3, 1) = \mathbf{S}(4) \oplus \mathbf{S}(3, 1) \oplus \mathbf{S}(2, 1^2) \oplus \mathbf{S}(2^2) \end{aligned}$$

at which point the product stabilises, i.e. for all $n \geq 4$, we have

$$\mathbf{S}(n-1, 1) \otimes \mathbf{S}(n-1, 1) = \mathbf{S}(n) \oplus \mathbf{S}(n-1, 1) \oplus \mathbf{S}(n-2, 1^2) \oplus \mathbf{S}(n-2, 2).$$

For $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{N}$, we let $P_k(n)$ denote the \mathbb{Z} -module with basis given by all set-partitions of $\{1, 2, \dots, k, \bar{1}, \bar{2}, \dots, \bar{k}\}$. A part of a set-partition is called a **block**. For example,

$$d = \{\{\bar{1}, \bar{2}, \bar{4}, 2, 5\}, \{\bar{3}\}, \{\bar{5}, \bar{6}, \bar{7}, 3, 4, 6, 7\}, \{\bar{8}, 8\}, \{1\}\},$$

is a set-partition (for $k = 8$) with 5 blocks and $p(d) = 3$. A set-partition can be represented by a diagram consisting of a frame with k distinguished points on the northern and southern boundaries, which we call vertices. We number the northern vertices from left to right by $\bar{1}, \bar{2}, \dots, \bar{k}$ and the southern vertices similarly by $1, 2, \dots, k$ and connect two vertices by a path

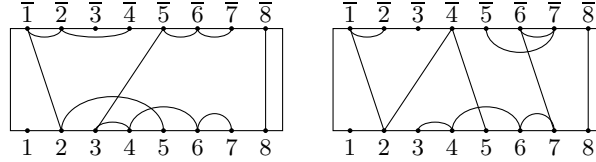


FIGURE 3. Two representatives of the set-partition d .

if they belong to the same block. Note that such a diagram is not uniquely defined, two diagrams representing the set-partition d above are given in Figure 3.

We define the product $x \cdot y$ of two diagrams x and y using the concatenation of x above y , where we identify the southern vertices of x with the northern vertices of y . If there are t connected components consisting only of middle vertices, then the product is set equal to n^t times the diagram with the middle components removed. Extending this by linearity defines a multiplication on $P_k(n)$.

We let $P_{k-1/2}(n)$ denote the subspace of $P_k(n)$ spanned by all set-partitions in which the integers k and \bar{k} appear in the same block. This subspace is clearly closed under the multiplication and therefore is a subalgebra. In [6], Enyang and Goodman showed that the sequence of algebras

$$P_0(n) \subseteq P_{\frac{1}{2}}(n) \subseteq P_1(n) \subseteq P_{\frac{3}{2}}(n) \subseteq P_2(n) \subseteq \dots$$

form a tower of diagram algebras obtained ‘by reflections’ from the (tower of) symmetric groups. The Jucys–Murphy elements for the partition algebras were defined in [11]; a recursive construction of these elements is given in [4]. Therefore these algebras fit into our framework and we have the following.

Proposition 5.1. Let λ be a partition of degree $r - s$, μ be a partition of degree s , and ν be a partition of degree less than or equal to r . We have that

$$\bar{g}_{\lambda, \mu}^{\nu} = \text{Hom}_{P_s^{\mathbb{Q}}(n)}(\Delta_s^{\mathbb{Q}}(\mu), \Delta_s^{\mathbb{Q}}(\nu \setminus \lambda)).$$

for n sufficiently large ($n \geq 2r$ will suffice).

Proof. This follows immediately from Theorem 4.4 and [1, Corollary 3.4]. \square

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